DIFFRACTION OF SURFACE WAVES AT A DOCK OF FINITE WIDTH

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We study the diffraction of small amplitude waves arriving from infinity to impinge on an immovable, rigid dock of finite width, situated on the surface of liquid of depth h. We use Jones' method to obtain the velocity potential. The normal and the oblique incidence of plane waves on the dock are considered. Coefficients of reflection and transition of the incoming waves are computed for the case of the normal incidence under the condition that the width of the dock is large compared with the depth of the liquid.

The plane problem of diffraction of the surface waves at a dock of finite width was dealt with by John [1] from the standpoint of the shallow water theory, and by Holford [2] for the waves on a liquid of infinite depth.

Several papers [3−6] deal with spatial wave motions of a liquid in the presence of a (rigid or elastic) dock occupying a half-plane of the free surface. In [7, 8] the case of a totally submerged dock is investigated. A variant of the Wiener-Hopf method [9] is used in [3−8] to solve boundary value problems which the velocity potential of the wave motion of the liquid must satisfy.

1. A dock forming a rigid obstacle of width $2a$ is situated on the surface of liquid and occupies the part of the free surface defined by $|x| < a, -\infty < z < \infty$. The coordinate origin is situated at the bottom and the y-axis points vertically upwards.

A plane wave $F^{(0)}(x, y, z, t) = D \operatorname{ch} C_0 y \operatorname{Re} e^{-i(kz - \varkappa x - \omega t)}$ $\quad (\varkappa^2 = C_0^2 - k^2)$ moving from infinity in the negative direction of the Ox-axis is incident upon the dock at a certain angle. Here $\pm iC_0$ are the roots of the equation $\beta \cos Ch + \sin Ch = 0$. The velocity potential $F(x, y, z, t)$ describing the motion of the liquid due to the incoming wave $F^{(0)}(x, y, z, t)$ should satisfy the Laplace equation $\Delta F(x, y, z, t) = 0$ in the region occupied by the liquid and the following boundary conditions:

$$\partial^2 F / \partial t^2 + g \partial F / \partial y = 0 \qquad \text{for } y = h, \ |x| > a, \ -\infty < z < \infty$$

$$\partial F / \partial y = 0 \qquad \text{for } y = h, \quad |x| < a, \ -\infty < z < \infty$$

$$\partial F / \partial y = 0 \qquad \text{for } y = 0, \ -\infty < x < \infty, \ -\infty < z < \infty$$

The motion of liquid should satisfy the corresponding conditions at infinity and near the edges of the dock $(\pm a, h)$. The latter condition is equivalent to the requirement that $\partial F / \partial t$ be bounded at the dock edges.

Function $F(x, y, z, t)$ is sought in the form

$$F(x, y, z, t) = \operatorname{Re} \{[\varphi(x, y) + D \operatorname{ch} C_0 y e^{-i\varkappa x}] e^{i(kz - \omega t)}\} \tag{1.1}$$

For $\varphi(x, y)$ we obtain

$$\Delta\varphi - k^2\varphi = 0 \qquad (0 \leqslant y \leqslant h, \ -\infty < x < \infty)$$

$$\partial\varphi / \partial y - \beta\varphi = 0 \qquad \text{for } y = h, \ |x| > a \quad (\beta = \omega^2 / g) \tag{1.2}$$

$$\partial\varphi / \partial y = -DC_0 \operatorname{sh} C_0 h e^{-i\varkappa x} \qquad \text{for } y = h, \ |x| < a \qquad \text{(cont.)}$$

$$\partial\varphi / \partial y = 0 \qquad \text{for } y = 0, \ -\infty < x < \infty$$

$$|\varphi(x, y)| < M = \text{const} \qquad \text{for } r = \sqrt{(x \mp a)^2 + (y - h)^2} \to 0$$

$$\lim_{x\to\infty} \varphi(x, y) = D_+ \operatorname{ch} C_0 y e^{i\vartheta_1 x}, \quad \lim_{x\to-\infty} \varphi(x, y) = D_- \operatorname{ch} C_0 y e^{-i\vartheta_2 x}$$

$$\vartheta_1 = \sigma_1 - i\tau_-, \qquad \vartheta_2 = \sigma_2 + i\tau_+, \qquad \tau_- < 0, \ \tau_+ > 0$$

In the final result τ_- and τ_+ both tend to zero.

Applying the Fourier transformations to the problem (1.2) and utilizing the notation

$$\Phi_+(\alpha, y) = \int_a^\infty \varphi(x, y) e^{i\alpha(x-a)} dx, \qquad \Phi_1(\alpha, y) = \int_{-a}^{a} \varphi(x, y) e^{i\alpha x} dx$$

$$\Phi_-(\alpha, y) = \int_{-\infty}^{-a} \varphi(x, y) e^{i\alpha(x+a)} dx, \qquad \gamma^2 = \alpha^2 + k^2$$

we obtain the following functional equation:

$$\Phi_+(\alpha) e^{i\alpha a} + \Phi_1(\alpha) K(\alpha) + e^{-i\alpha a}\Phi_-(\alpha) = -\frac{2DC_0 \operatorname{sh} C_0 h \sin(\alpha - \varkappa) a \operatorname{ch} \gamma h}{(\gamma \operatorname{sh} \gamma h - \beta \operatorname{ch} \gamma h)(\alpha - \varkappa)} \tag{1.3}$$

$$\alpha = \sigma + i\tau, \ \tau_- < \tau < \tau_+, \ -\infty < \sigma < \infty$$

Here $\Phi_1(\alpha)$ is an entire function, $K(\alpha)$ is regular, $\Phi_+(\alpha)$ is regular in the semiplane $\tau > \tau_-$, and $\Phi_-(\alpha)$ is regular in the semiplane $\tau < \tau_+$

$$K(\alpha) = \frac{\gamma \operatorname{sh} \gamma h}{\gamma \operatorname{sh} \gamma h - \beta \operatorname{ch} \gamma h}$$

$$\Phi(\alpha, y) = \left\{ \frac{\beta}{\gamma \operatorname{sh} \gamma h} [e^{i\alpha a}\Phi_+(\alpha) + e^{-i\alpha a}\Phi_-(\alpha)] - \frac{2DC_0 \operatorname{sh} C_0 h \sin(\alpha - \varkappa) a}{\gamma \operatorname{sh} \gamma h (\alpha - \varkappa)} \right\} \operatorname{ch} \gamma y \tag{1.4}$$

2. We use the approximate Wiener-Hopf method [10] to solve (1.3).

We factorize [3–10] the function $K(\alpha) = K_+(\alpha) K_-(\alpha)$

$$K_+(\alpha) = i \left(\frac{h}{\beta}\right)^{1/2} \frac{k - i\alpha}{\delta + \alpha} \frac{\rho_0}{h} \prod_{n=1}^{\infty} \frac{(1 + h^2k^2 / n^2\pi^2)^{1/2} - i\alpha h / n\pi}{(1 + h^2k^2 / \rho_n)^{1/2} - i\alpha h / \rho_n}$$

$$\rho_0^2 = h^2(\delta^2 + k^2), \quad K_-(\alpha) = K_+(-\alpha)$$

Here $\pm \rho_0/h$ and $\pm i\rho_n/h$ are the roots of the equation $\rho \operatorname{sh} \rho h - \beta \operatorname{ch} \rho h = 0$, and $\rho_n = n\pi + \beta h / n\pi$ for $n \gg 1$.

Multiplying (1.3) by $[e^{i\alpha a} K_+(\alpha)]^{-1}$ and by $[e^{-i\alpha a} K_-(\alpha)]^{-1}$, and using

$$\Phi_1(\alpha) e^{-i\alpha a} = S_-(\alpha), \quad \Phi_1(\alpha) e^{i\alpha a} = S_+(\alpha)$$

we obtain the following two equations

$$\frac{\Phi_\pm}{K_\pm} + S_\mp K_\mp + e^{\mp 2i\alpha a} \frac{\Phi_\mp}{K_\pm} = -\frac{2DC_0 \operatorname{sh} C_0 h e^{\mp i\alpha a} \sin(\alpha - \varkappa) a \operatorname{ch} \gamma h}{K_\pm (\gamma \operatorname{sh} \gamma h - \beta \operatorname{ch} \gamma h)(\alpha - \varkappa)}$$

One equation is obtained by taking the upper sign, and the second equation by taking the lower sign. The first equation applies within the limits $\tau_- < \tau < 0$, and the second equation within the limits $0 < \tau < \tau_+$.

Taking into account the edge conditions (1.2) and applying the partition theorem

([9] Sect. 1, 3) together with the Liouville theorem to the above equations, we obtain two following integral equations:

$$\frac{\Phi_\pm(\alpha)}{K_\pm(\alpha)} \pm \frac{1}{2\pi i} \int\limits_{-\infty+ic_\mp}^{\infty+ic_\mp} \frac{e^{\mp 2i\xi a}\Phi_\mp(\xi)}{K_\pm(\xi)(\xi-\alpha)} d\xi = \tag{2.1}$$

$$= \mp \frac{DC_0 \operatorname{sh} C_0 h}{\pi i} \int\limits_{-\infty+if_\mp}^{\infty+if_\mp} \frac{e^{\mp i\xi a} \sin(\xi-\varkappa) a \operatorname{ch} \eta h d\xi}{K_\pm(\xi)[\eta \operatorname{sh} \eta h - \beta \operatorname{ch} \eta h](\xi-\varkappa)(\xi-\alpha)}$$

$$\tau_- < c_- < \tau < 0, \ \tau_- < f_- < \tau < 0, \ 0 < \tau < c_+ < \tau_+, \ 0 < \tau < f_+ < \tau_+$$

$$\eta^2 = \xi^2 + k^2$$

Choosing l and m from $(0, \tau_+)$, setting $C_+ = -C_- = l$, $f_+ = f_- = m$, replacing ξ by $-\xi$ and α by $-\alpha$ in the relevant equations of (2.1), we obtain

$$\frac{\Phi_\pm(\pm\alpha)}{K_+(\alpha)} - \frac{1}{2\pi i} \int\limits_{il-\infty}^{il+\infty} \frac{e^{2i\xi a}\Phi_\mp(\mp\xi)}{K_-(\xi)(\xi+\alpha)} d\xi = \tag{2.2}$$

$$= \frac{DC_0 \operatorname{sh} C_0 h}{\pi i} \int\limits_{im-\infty}^{im+\infty} \frac{e^{i\xi a} \sin(\xi \pm \varkappa) a \operatorname{ch} \eta h d\xi}{K_-(\xi)(\eta \operatorname{sh} \eta h - \beta \operatorname{ch} \eta h)(\xi \pm \varkappa)(\xi+\alpha)}$$

In both of these equations $\tau > \sup(-l, -m)$. We next introduce the functions

$$G_+^+(\alpha) = \Phi_+(\alpha) - \Phi_-(-\alpha), \quad G_+^-(\alpha) = \Phi_+(\alpha) + \Phi_-(-\alpha) \tag{2.3}$$

which are, in accordance with (2.2), the solutions of integral equations

$$\frac{G_+^\lambda(\alpha)}{K_+(\alpha)} \pm \frac{1}{2\pi i} \int\limits_{il-\infty}^{il+\infty} \frac{e^{2i\xi a} G_+^\lambda(\xi)}{K_-(\xi)(\xi+\alpha)} d\xi = \tag{2.4}$$

$$= \frac{DC_0 \operatorname{sh} C_0 h}{\pi i} \int\limits_{im-\infty}^{im+\infty} \frac{e^{i\xi a} \operatorname{ch} \eta h}{K_-(\xi)(\eta \operatorname{sh} \eta h - \beta \operatorname{ch} \eta h)(\xi+\alpha)} \left[\frac{\sin(\xi+\varkappa) a}{\xi+\varkappa} \mp \frac{\sin(\xi-\varkappa) a}{\xi-\varkappa}\right] d\xi$$

In (2.4)and the following expressions the symbol λ assumes respectively (+) or (−) sign. Taking into account that

$$K_+(\alpha) K_-(\alpha) = \frac{\gamma \operatorname{sh} \gamma h}{\gamma \operatorname{sh} \gamma h - \beta \operatorname{ch} \gamma h}$$

we can write (2.4) as

$$\frac{G_+^\lambda(\alpha)}{K_+(\alpha)} \pm \frac{1}{2\pi i} \int\limits_{il-\infty}^{il+\infty} \frac{G_+^\lambda(\xi) e^{2i\xi a} [\eta \operatorname{sh} \eta h - \beta \operatorname{ch} \eta h] K_+(\xi)}{\eta \operatorname{sh} \eta h (\xi+\alpha)} d\xi = \tag{2.5}$$

$$= \frac{DC_0 \operatorname{sh} C_0 h}{\pi i} \int\limits_{im-\infty}^{im+\infty} \frac{e^{i\xi a} \operatorname{ch} \eta h K_+(\xi)}{\eta \operatorname{sh} \eta h (\xi+\alpha)} \left[\frac{\sin(\xi+\varkappa) a}{\xi+\varkappa} \mp \frac{\sin(\xi-\varkappa) a}{\xi-\varkappa}\right] d\xi$$

The singularities of the integrand functions are poles

$$\xi = ik, \ i\mu_n, \ \mu_n = [k^2 + n^2\pi^2 / h^2]^{1/2} \quad (n = 1, 2, \ldots)$$

Using the residue theory, we can write (2.5) in the form

$$\frac{G_+^\lambda(\alpha)}{K_+(\alpha)} \pm \frac{i\beta}{2kh} \frac{G_+^\lambda(ik)\, e^{-2ka} K_+(ik)}{ik+\alpha} \pm \frac{i\beta}{h} \sum_{n=1}^{\infty} \frac{G_+^\lambda(i\mu_n)\, e^{-2\mu_n a} K_+(i\mu_n)}{\mu_n(i\mu_n+\alpha)} =$$

$$= -\frac{i2DC_0 \operatorname{sh} C_0 h}{h} \left\{ \frac{1}{2k} \frac{K_+(ik)}{ik+\alpha} \left[\frac{\sin(ik+\varkappa)a}{ik+\varkappa} \mp \frac{\sin(ik-\varkappa)a}{ik-\varkappa} \right] e^{-ka} + \right.$$

$$\left. + \sum_{n=1}^{\infty} \frac{K_+(i\mu_n)\, e^{-\mu_n a}}{\mu_n(i\mu_n+\alpha)} \left[\frac{\sin(i\mu_n+\varkappa)a}{i\mu_n+\varkappa} \mp \frac{\sin(i\mu_n-\varkappa)a}{i\mu_n-\varkappa} \right] \right\} \tag{2.6}$$

Setting now $\alpha = ik$, $i\mu_j$ $(j = 1, 2, ..)$, in (2.6), we obtain at these points the following infinite algebraic systems for $G_+^\lambda(\alpha)$:

$$G_+^\lambda(ik) = \left[\frac{1}{K_+(ik)} \pm \frac{\beta K_+(ik)}{4k^2 h} e^{-2ka} \right]^{-1} \left\{ \mp \frac{\beta}{h} \sum_{n=1}^{\infty} \frac{e^{-2\mu_n a} G_+^\lambda(i\mu_n) K_+(i\mu_n)}{\mu_n(\mu_n+k)} - \right.$$

$$- \frac{2DC_0 \operatorname{sh} C_0 h}{h} \left[\frac{1}{2k} \frac{K_+(ik)}{2k} \left(\frac{\sin(ik+\varkappa)a}{ik+\varkappa} \mp \frac{\sin(ik-\varkappa)a}{ik-\varkappa} \right) e^{-ka} + \right.$$

$$\left.\left. + \sum_{n=1}^{\infty} \frac{K_+(i\mu_n)\, e^{-\mu_n a}}{\mu_n(\mu_n+k)} \left(\frac{\sin(i\mu_n+\varkappa)a}{i\mu_n+\varkappa} \mp \frac{\sin(i\mu_n-\varkappa)a}{i\mu_n-\varkappa} \right) \right] \right\}$$

$$\frac{G_+^\lambda(i\mu_j)}{K_+(i\mu_j)} \pm \frac{\beta K_+(ik)}{2kh} \frac{G_+^\lambda(ik)\, e^{-2ka}}{k+\mu_j} \pm \frac{\beta}{h} \sum_{n=1}^{\infty} \frac{G_+^\lambda(i\mu_n)\, e^{-2\mu_n a} K_+(i\mu_n)}{\mu_n(\mu_n+\mu_j)} =$$

$$= -\frac{2DC_0 \operatorname{sh} C_0 h}{h} \left\{ \frac{1}{2k} \frac{K_+(ik)}{k+\mu_j} \left[\frac{\sin(ik+\varkappa)a}{ik+\varkappa} \mp \frac{\sin(ik-\varkappa)a}{ik-\varkappa} \right] e^{-ka} + \right.$$

$$\left. + \sum_{n=1}^{\infty} \frac{K_+(i\mu_n)\, e^{-\mu_n a}}{\mu_n(\mu_n+\mu_j)} \left[\frac{\sin(i\mu_n+\varkappa)a}{i\mu_n+\varkappa} \mp \frac{\sin(i\mu_n-\varkappa)a}{i\mu_n-\varkappa} \right] \right\}$$

$$(j = 1, 2, \ldots) \tag{2.7}$$

which are fully regular, when

$$\sum_{n=1}^{\infty} \left| \frac{\beta}{h} \frac{K_+(i\mu_n)}{\mu_n} e^{-2\mu_n a} K_+(i\mu_j) \left\{ \pm \frac{1}{\mu_n+\mu_j} - \right.\right. \tag{2.8}$$

$$\left.\left. - \frac{2k\beta K_+^2(ik)}{[4k^2 h e^{2ka} \pm \beta K_+^2(ik)]\,(k+\mu_n)(k+\mu_j)} \right\} \right| < 1 \qquad (j = 1, 2, \ldots)$$

i.e. provided that the sum of the moduli of the coefficients in each row is less than unity [11].

The relation (2.8) can be regarded as the condition defining the values of a / h, for which the systems (2.7) are regular, i.e. limiting value of the ratio a / h up to which the systems are fully regular can be obtained for each particular case.

When the conditions (2.8) hold, the systems (2.7) are fully regular and their free terms are bounded. Consequently they have bounded solutions which can be obtained using the method of consecutive approximations [11].

Having solved (2.7) and using (2.3), we can find $\Phi_+(\alpha)$ and $\Phi_-(\alpha)$.

3. Let us limit ourselves to the case $2a / h \gg 1$, i.e. assume that the width of the dock exceeds the depth of the liquid. The (2.6) yields the following expression for $G_+^\lambda(\alpha)$:

$$G_+^\lambda(\alpha) = -iK_+(\alpha)\left\{\frac{e^{-ka}}{kh}\frac{K_+(ik)}{ik+\alpha}\left[\pm\frac{\beta}{2}G_+^\lambda(ik)e^{-ka}+DC_0\operatorname{sh}C_0h\left(\frac{\sin(ik+\varkappa)a}{ik+\varkappa}\mp\right.\right.\right.$$

$$\left.\left.\left.\mp\frac{\sin(ik-\varkappa)a}{ik-\varkappa}\right)\right]+i\frac{2DC_0\operatorname{sh}C_0h}{h}\sum_{n=1}^{\infty}\frac{K_+(i\mu_n)}{\mu_n(i\mu_n+\alpha)}\left[\frac{e^{-\varkappa a}}{i\mu_n+\varkappa}\mp\frac{e^{i\varkappa a}}{i\mu_n-\varkappa}\right]\right\} \quad (3.1)$$

In this case (2.7) yields the following expression for $G_+^\lambda(ik)$:

$$G_+^\lambda(ik) = -\frac{2DC_0\operatorname{sh}C_0h}{h}\left[\frac{1}{K_+(ik)}\pm\frac{\beta e^{-2ka}}{4k^2h}K_+(ik)\right]^{-1}\left\{\frac{K_+(ik)}{k^2}\left[\frac{\sin(ik+\varkappa)a}{ik+\varkappa}\mp\right.\right.$$

$$\left.\left.\mp\frac{\sin(ik-\varkappa)a}{ik-\varkappa}\right]e^{-ka}+\sum_{n=1}^{\infty}\frac{K_+(i\mu_n)}{\mu_n(\mu_n+k)}\left[\frac{e^{-i\varkappa a}}{i\mu_n+\varkappa}\mp\frac{e^{i\varkappa a}}{i\mu_n-\varkappa}\right]\right\}$$

Then, taking into account (2.3) we obtain the following expressions for $\Phi_+(\alpha)$ and $\Phi_-(\alpha)$:

$$\Phi_+(\alpha) = -iK_+(\alpha)\left\{\frac{e^{-ka}}{kh}\frac{K_+(ik)}{ik+\alpha}\left[\frac{\beta}{2}\frac{G_+^+(ik)-G_+^-(ik)}{2}e^{-ka}+\right.\right.$$

$$\left.\left.+DC_0\operatorname{sh}C_0h\frac{\sin(ik-\varkappa)a}{ik-\varkappa}\right]+\frac{iDC_0\operatorname{sh}C_0h}{h}\sum_{n=1}^{\infty}\frac{K_+(i\mu_n)}{\mu_n(i\mu_n+\alpha)}\frac{e^{-i\varkappa a}}{i\mu_n+\varkappa}\right\} \quad (3.2)$$

$$\Phi_-(\alpha) = -iK_-(\alpha)\left\{\frac{e^{-ka}}{kh}\frac{K_+(ik)}{ik-\alpha}\left[-\frac{\beta}{2}\frac{G_+^+(ik)+G_+^-(ik)}{2}e^{-ka}+\right.\right.$$

$$\left.\left.+DC_0\operatorname{sh}C_0h\frac{\sin(ik-\varkappa)a}{ik-\varkappa}\right]+\frac{iDC_0\operatorname{sh}C_0h}{h}\sum_{n=1}^{\infty}\frac{K_+(i\mu_n)}{\mu_n(i\mu_n-\alpha)}\frac{e^{i\varkappa a}}{i\mu_n-\varkappa}\right\} \quad (3.3)$$

Applying the inverse Fourier transformation to (1.4) and taking into account (3.2) and (3.3), we obtain the solution of the problem (1.2) in the form

1) for $x < -a$

$$\varphi(x, y) = (a_0+ib_0)\operatorname{ch}C_0ye^{-i\delta x}\left\{\frac{e^{-ka}}{kh}\frac{K_+(ik)}{ik-\delta}\left[\frac{\beta}{2}\frac{G_+^+(ik)+G_+^-(ik)}{2}-\right.\right.$$

$$\left.\left.-DC_0\operatorname{sh}C_0h\frac{\sin(ik-\varkappa)a}{ik-\varkappa}\right]-\frac{iDC_0\operatorname{sh}C_0h}{h}\sum_{n=1}^{\infty}\frac{K_+(i\mu_n)}{\mu_n(i\mu_n-\delta)}\frac{e^{i\varkappa a}}{i\mu_n-\varkappa}\right\}-$$

$$-\sum_{p=1}^{\infty}(a_p+ib_p)e^{\nu_p x}\cos\frac{\rho_p y}{h}\left\{\frac{e^{-ka}}{kh}\frac{K_+(ik)}{k-\nu_p}\left[\frac{\beta}{2}\frac{G_+^+(ik)+G_+^-(ik)}{2}-\right.\right.$$

$$\left.\left.-DC_0\operatorname{sh}C_0h\frac{\sin(ik-\varkappa)a}{ik-\varkappa}\right]-\frac{iDC_0\operatorname{sh}C_0h}{h}\sum_{n=1}^{\infty}\frac{K_+(i\mu_n)}{\mu_n(\mu_n-\nu_p)}\frac{e^{i\varkappa a}}{i\mu_n-\varkappa}\right\} \quad (3.4)$$

2) for $|x| < a$

$$\varphi(x, y) = 2DC_0h\operatorname{sh}C_0h\sum_{n=1}^{\infty}\frac{(-1)^n e^{-\mu_n a}}{\mu_n(n^2\pi^2+C_0^2h^2)}[\mu_n\cos(i\mu_n x+\varkappa a)-$$

$$-\varkappa\sin(i\mu_n x+\varkappa a)]\cos\frac{n\pi y}{h}+\frac{e^{-ka}}{kh}\left\{\frac{\beta}{2}[G_+^+(ik)\operatorname{sh}kx+G_+^-(ik)\operatorname{ch}kx]+\right.$$

$$\left.+\frac{D\operatorname{sh}C_0h}{C_0}[k\cos(ikx+\varkappa a)-\varkappa\sin(ikx+\varkappa a)]\right\}-De^{-i\varkappa x}\operatorname{ch}C_0y \quad (3.5)$$

3) for $x > a$

$$\varphi(x, y) = (a_0 + ib_0)\,\mathrm{ch}\, C_0 y e^{i\delta x}\left\{\frac{e^{-ka}}{kh}\left[\frac{\beta}{2}\,\frac{G_+^{-}(ik) - G_+^{+}(ik)}{2} - \right.\right.$$

$$\left.\left. - DC_0\,\mathrm{sh}\, C_0 h\,\frac{\sin(ik+\varkappa)\,a}{ik+\varkappa}\right] - \frac{iDC_0\,\mathrm{sh}\, C_0 h}{h}\sum_{n=1}^{\infty}\frac{K_+(i\mu_n)}{\mu_n(i\mu_n + k)}\,\frac{e^{-i\varkappa\alpha}}{i\mu_n + \delta}\right\} -$$

$$- \sum_{p=1}^{\infty}(a_p + ib_p)\,e^{-\nu_p x}\cos\frac{\rho_p y}{h}\left\{\frac{e^{-ka}}{kh}\,\frac{K_+(ik)}{k - \nu_p}\left[\frac{\beta}{2}\,\frac{G_+^{-}(ik) - G_+^{+}(ik)}{2} - \right.\right.$$

$$\left.\left. - DC_0\,\mathrm{sh}\, C_0 h\,\frac{\sin(ik+\varkappa)\,a}{ik+\varkappa}\right] - \frac{iDC_0\,\mathrm{sh}\, C_0 h}{h}\sum_{n=1}^{\infty}\frac{K_+(i\mu_n)}{\mu_n(\mu_n - \nu_p)}\,\frac{e^{-i\varkappa a}}{i\mu_n + \varkappa}\right\} \qquad (3.6)$$

Here

$$\nu_p = [k^2 + \rho_p^2 / h^2]^{1/2}, \quad a_0 + ib_0 = \frac{\beta\rho_0^2 e^{-i\delta a}}{K_+(\delta)\,\delta h\,(\beta^2 h^2 - \beta h - \rho_0^2)\,\mathrm{ch}\,\rho_0}$$

$$a_p + ib_p = \frac{\beta\rho_p^2 e^{\nu_p a}}{K_+(i\nu_p)\,\nu_p h\,(\beta h - \beta^2 h^2 - \rho_p^2)\cos\rho_p}$$

Let now τ_- and τ_+ both tend to zero. Taking into account (3.4), (3.5), (3.6) and (1.1) we obtain the final expression for the velocity potential, respectively, for the departed wave, for the motion of the liquid under the dock, and for the reflected wave.

4. The case of normal incidence of the waves upon the dock ($k = 0$) must be considered separately for the following reason. The passage to the limit $k \to 0$ is not possible in the solution obtained because for $\alpha = 0$, a double root appears in the kernel

$$K(\alpha) = \frac{\alpha\,\mathrm{sh}\,\alpha h}{\alpha\,\mathrm{sh}\,\alpha h - \beta\,\mathrm{ch}\,\alpha h}$$

of the functional equation (1.3). This violates the regularity of $K(\alpha)$ on the strip $\tau_- < \tau < \tau_+$, $-\infty < \sigma < \infty$ and $K(\alpha) = K_+(\alpha)K_-(\alpha)$ cannot be factorized in such a manner that $K_+(\alpha)$ and $K_-(\alpha)$ are both regular and have no zeros within this strip. We can overcome this difficulty as follows. The kernel $K(\alpha)$ accompanies on the strip defined above the unknown entire function $\Phi_1(\alpha)$ as its coefficient; we can therefore relate α^2 to $\Phi_1(\alpha)$ and write the functional equation (1.3) for this case in the form

$$e^{i\alpha a}\Phi_+(\alpha) + \Phi_1^*(\alpha)\,\frac{\mathrm{sh}\,\alpha h}{\alpha(\alpha\,\mathrm{sh}\,\alpha h - \beta\,\mathrm{ch}\,\alpha h)} + e^{-i\alpha a}\Phi_-(\alpha) =$$

$$= -\frac{2DC_0\,\mathrm{sh}\, C_0 h \sin(\alpha - \varkappa)\,a\,\mathrm{ch}\,\alpha h}{(\alpha\,\mathrm{sh}\,\alpha h - \beta\,\mathrm{ch}\,\alpha h)(\alpha - \varkappa)}, \qquad \Phi_1^*(\alpha) = \alpha^2\Phi_1(\alpha)$$

Factorization of $K^*(\alpha)$ yields

$$K^*(\alpha) = \frac{\mathrm{sh}\,\alpha h}{\alpha(\alpha\,\mathrm{sh}\,\alpha h - \beta\,\mathrm{ch}\,\alpha h)} = K_+^*(\alpha)\,K_-^*(\alpha)$$

$$K_+^*(\alpha) = \left(\frac{h}{\beta}\right)^{1/2}\frac{i}{\delta + \alpha}\,\frac{\rho_0}{h}\prod_{n=1}^{\infty}\frac{1 - i\alpha h / n\pi}{1 - i\alpha h / \rho_n}$$

$$K_-^*(\alpha) = K_+^*(-\alpha), \quad \rho_0 / h = C_0 = \varkappa = \delta$$

$\pm\rho_0 / h$ and $\pm i\rho_n / h$ are roots $\rho\,\mathrm{th}\rho h = \beta$

Equations (2.1) can now be written as

$$\frac{\Phi_\pm(\alpha)}{K_\pm^*(\alpha)} \pm \frac{1}{2\pi i} \int\limits_{-\infty+ic_\mp}^{\infty+ic_\mp} \frac{e^{\mp 2i\xi a}\Phi_\mp(\xi)}{K_\pm^*(\xi)(\xi-\alpha)} d\xi \pm$$

$$\pm \frac{DC_0 \operatorname{sh} C_0 h}{\pi i} \int\limits_{-\infty+if_\mp}^{\infty+if_\mp} \frac{e^{\mp i\xi a} \sin(\xi - \varkappa) a \operatorname{ch} \xi h d\xi}{K_\pm^*(\xi)(\xi \operatorname{sh} \xi h - \beta \operatorname{ch} \xi h)(\xi - \varkappa)(\xi - \alpha)} = B_\pm$$

where B_+ and B_- are constants to be determined from the boundary conditions [9].

Assuming $2a / h \gg 1$, we obtain the following expression for $G_+^\lambda(\alpha)$ for $k = 0$:

$$G_+^\lambda(\alpha) = K_+^*(\alpha)\Big\{B_+ \pm B_- - DC_0 \operatorname{sh} C_0 h \sum_{n=1}^{\infty} \frac{n\pi}{in\pi + \alpha h} K_+^*\left(\frac{in\pi}{h}\right) \times$$

$$\times \left[\frac{e^{-i\varkappa a}}{in\pi + \varkappa h} \mp \frac{e^{i\varkappa a}}{in\pi - \varkappa h}\right]\Big\}$$

Repeating the multiplications performed previously, we obtain the required solution in the form

$$F^{(1)}(x, y, t) = |(T_0 + iR_0)\psi_+(-\delta)| \operatorname{ch} C_0 y \cos(\varkappa x - \omega t) +$$

$$+ \sum_{p=1}^{\infty} |(T_p + iR_p)\psi_+(-i\nu_p)| \cos \nu_p y \cos \omega t$$

$$F^{(2)}(x, y, t) =$$

$$= \Big\{\Big| D \operatorname{sh} C_0 h \frac{i\varkappa a + 1}{\varkappa h} e^{-i\varkappa a} - i \frac{K_-^*(0)[\psi_+'(0) + ia\psi_+(0)] - \psi_+(0) K_-^{*\prime}(0)}{K_-^{*2}(0)}\Big| +$$

$$+ \beta \sum_{p=1}^{\infty} \frac{(-1)^p}{p\pi} \exp \frac{-p\pi a}{h} \cos \frac{p\pi y}{h} \left| K_+^*\left(\frac{ip\pi}{h}\right)\right| \Big[\left|\psi_+\left(\frac{ip\pi}{h}\right)\right| \exp \frac{p\pi x}{h} +$$

$$+ \left|\psi_-\left(-\frac{ip\pi}{h}\right)\right| \exp \frac{-p\pi x}{h}\Big] - x \left|\frac{\psi_+(0)}{K_-^*(0)}\right| -$$

$$- 2DC_0 h \operatorname{sh} C_0 h \sum_{n=1}^{\infty} \frac{(-1)^n}{n\pi} \exp \frac{-n\pi a}{h} \cos \frac{n\pi y}{h} \frac{\operatorname{ch}(n\pi x / h)}{\sqrt{n^2\pi^2 + \varkappa^2 h^2}}\Big\} \cos \omega t$$

$$F^{(3)}(x, y, t) = D\left|1 + \frac{(T_0 + iR_0)\psi_-(\delta)}{D}\right| \operatorname{ch} C_0 y \cos(\varkappa x + \omega t) +$$

$$+ \sum_{p=1}^{\infty} |(T_p + iR_p)\psi_-(i\nu_p)| e^{\nu_p x} \cos \nu_p y \cos \omega t$$

where

$$T_0 + iR_0 = \frac{i\beta\rho_0 e^{-i\delta a}}{\delta^3(\beta h - \beta^2 h^2 + \rho_0^2) \operatorname{ch} \rho_0 K_+^*(\delta)}, \quad \nu_p = \rho_p / h$$

$$T_p + iR_p = \frac{\beta\rho_p e^{\nu_p a}}{\nu_p^2(\beta h - \beta^2 h^2 - \rho_p^2) \cos \rho_p K_+^*(i\nu_p)}, \quad \psi_+(\alpha) = \frac{\Phi_+}{K_+^*}, \quad \psi_-(\alpha) = \frac{\Phi_-}{K_-^*}$$

$$B_+ = i \frac{\psi_+'(0) + \psi_-'(0)}{2a} + DC_0 \operatorname{sh} C_0 h \sum_{n=1}^{\infty} \frac{e^{-i\varkappa a}}{i\varkappa h - n\pi} K_+^*\left(\frac{in\pi}{h}\right) -$$

$$- \frac{[aK_+^*(0) - iK_+^*(0)]}{ah} D \operatorname{sh} C_0 h \sin \varkappa a - \frac{iD \operatorname{sh} C_0 h K_+^{*\prime}(0)}{2ah}\left(a \cos \varkappa a - \frac{\sin \varkappa a}{\varkappa}\right)$$

$$B_{-} = D \operatorname{sh} C_0 h \left\{ C_0 \sum_{n=1}^{\infty} K_{+}^{*} \left(\frac{in\pi}{h} \right) \left[\frac{e^{-i\varkappa a}}{i\varkappa h - n\pi} - \frac{e^{i\varkappa a}}{i\varkappa h + n\pi} \right] - 2K_{+}^{*}(0) \frac{\sin \varkappa a}{h} \right\} - B_{+}$$

The coefficients of reflection and transition of the waves are given by the formulas [12]

$$f_r = \frac{|F^{(1)}|}{|F^{(0)}|} = \frac{|(T_0 + iR_0)\psi_{+}(-\delta)|}{D}, \quad f_t = \frac{|F^{(3)}|}{|F^{(0)}|} = \left| \frac{(T_0 + iR_0)\psi_{-}(\delta)}{D} + 1 \right|$$

where $|F^{(1)}|$, $|F^{(3)}|$ and $|F^{(0)}|$ denote the amplitudes of the reflected, departed and incident waves, respectively.

The pressure of liquid under the dock is given by

$$p(x, y, t) = -\rho \frac{\partial F^{(2)}(x, y, t)}{\partial t} - \rho g y$$

Computation (in MKS system) of the pressure distribution along the x-axis for the parameter values

$D = 0.25$ m²/sec , $h = 1$ m , $a = 2$ m , $\omega = 4.34$ sec^{-1}, $\varkappa = 2$ m^{-1}, $t = 1$ sec

gave the following results:

$x =$	-2	-1	0	1	2
$p =$	-50	0	67	144	230

These data indicate that the pressure change is larger near the right hand side edge, i.e. the dock dampens the waves. In this case $f_t = 0.5$ and $f_r = 0.86$. It should be noted that according to the results of [1] the pressure under the dock varies linearly, while the pressure distribution investigated in the present paper is nonlinear. This is explained by the fact that in the present paper we consider the hydrodynamic pressure as opposed to the hydrostatic pressure in [1].

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BIBLIOGRAPHY

1. John, F., On the motion of floating bodies. Comm. Pure Appl. Math. Vol. 2, №1, 1949.
2. Holford, R. L., Short surface waves in the presence of a finite dock. I, II. Proc. Cambridge Philos. Soc., Vol. 60, №4, 1964.
3. Heins, A. E., Water waves over a channel of finite depth with a dock. Amer. J. Math., Vol. 70, №4, 1948.
4. Weitz, M. and Keller, J. B., Reflection of water waves from floating ice in water of finite depth. Comm. Pure Appl. Math., Vol. 3, №3, 1950.
5. Heins, A. E., Some remarks on the coupling of two ducts. J. Math. Phys., Vol. 30, №3, 1951.
6. Heins, A. E., The scope and limitation of the method of Wiener and Hopf. Comm. Pure Appl. Math., Vol. 9, №3, 1956.
7. Heins, A. E., Water waves over a channel of finite depth with a submerged plane barrier. Quart. Appl. Math., Vol. 11, №2, 1953.
8. Greene, T. R. and Heins, A. E., Water waves over a channel of infinite depth. Quart. Appl. Math., 11, 201-214, 1953.
9. Noble, B., Methods Based on the Wiener-Hopf Technique for the Solution of Partial Differential Equations. Pergamon Press, 1958.

10. Jones, D. S., A simplifying technique in the solution of a class of diffraction problems. Quart. J. Math. Ser. 2, Vol. 3, №1, 1952.
11. Kantorovich, L. V. and Krylov, V. N., Approximate methods in Higher Analysis, 3rd ed. M.-L., Gostekhizdat, 1949.
12. Stoker, J. J., Water Waves. Interscience, 1957.

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THE PROPAGATION OF SMALL PERTURBATIONS IN A VISCOELASTIC FLUID

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The propagation of small-amplitude waves is investigated in an incompressible viscoelastic fluid, the rheological behavior of which is described by a nonlinear differential-operator equation of state.

Waves in a linear viscoelastic medium have been discussed in detail in [1, 2]. In this paper, we consider the models of Oldroyd [3] and de Witt [4], and the generalizations of these models for the case of the finite spectrum of the times of relaxation and retardation. For the stated models an invariant formulation is adduced for the conditions of evolutionarity of a system of hydrodynamic equations. Possible types of short transverse waves are esteblished for media which possess transient elasticity. The phase polars and group polars of a point source are considered. The local characteristics are adduced for high-frequency transverse waves in the case of reflection and refraction at the boundary of an Oldroyd fluid with a linearly elastic solid.

Small perturbations are considered for the presence in the fluid of a stressed state which is different from the hydrostatic pressure.

1. Formulation of the conditions for evolutionarity of the hydrodynamic equations of models possessing a finite set of relaxation and retardation times. The system of dynamic equations for an incompressible viscoelastic fluid consists of the equation of continuity

$$\operatorname{div} \mathbf{v} = 0 \tag{1.1}$$

the equations of momenta

$$\rho\, d\mathbf{v}/dt = -\nabla p + \operatorname{div} \mathbf{T} + \rho \mathbf{F} \tag{1.2}$$

and the tensor equation of state

$$P_r\left(\frac{D}{Dt}\right)\mathbf{T} = 2\eta Q_s\left(\frac{D}{Dt}\right)\mathbf{E} \tag{1.3}$$

In Eq. (1.3), $\mathbf{T}$ is the tensor of "viscoelastic" stresses, $\mathbf{E}$ is the tensor of the rate of deformation; $P_r(D\,/\,Dt)$ and $Q_s\,(D\,/\,Dt)$ are differential operators representing polynomials of D/Dt

$$P_r\left(\frac{D}{Dt}\right) = \prod_{i=1}^{r}\left(1 + \lambda_i \frac{D}{Dt}\right), \quad Q_s\left(\frac{D}{Dt}\right) = \prod_{i=1}^{s}\left(1 + \theta_i \frac{D}{Dt}\right) \tag{1.4}$$

The quantities λ_i and θ_i form relaxation and retardation spectra, respectively. The symbol $D\mathbf{A}/Dt$ denotes the relative convective derivative of the tensor $\mathbf{A}$ defined by